

APPROXIMATE EQUILIBRIUM ASSET PRICES

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Abstract. Arguing that total consumer wealth is unobservable, we invert the (approximate) consumption function to reconstruct, in a world with Kreps-Porteus generalized isoelastic preferences, i) the wealth that supports the agents' observed consumption as an optimal outcome and ii) the rate of return on the consumers' wealth portfolio. This allows us to (approximately) price assets solely as a function of their payoffs and of consumption — in both homoskedastic or heteroskedastic environments. We compare implied equilibrium returns on the wealth portfolio to observed stock market returns and gauge whether the stock market is a good proxy for unobserved aggregate wealth.

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1. Introduction

This paper is motivated by two observations. The first one is empirical. According to Gutter (1999), non-human wealth represented less than 10% of total household wealth in the United States in 1998. Moreover, financial assets amounted at the same date, at market value, to only 15% of total household wealth. The Survey of Consumer Finances shows that, in that same year, direct stock holdings represented only 10% of total financial assets. Including mutual fund shares and life insurance, this proportion only climbs to slightly more than 15% of total financial assets or 20% of total household wealth. Even if one excludes human capital and inside financial assets (such as banks' deposits and public debt), there is therefore much more to consumer wealth than stocks and much more to the rate of return on wealth than the rate of return on the stock market. The rate of return on the stock market—the measure of the rate of return on wealth used by most of the capital asset pricing literature—can only be an imperfect proxy for the rate of return on wealth.

The second observation pertains to theory. Many authors seem to have forgotten that two of the main contenders in the search for the explanation of excess returns—the static (or market) capital asset pricing model (SCAPM) and the

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consumption capital asset pricing model (CCAPM)—are not independent and unrelated models. Regardless of the view one takes on the exact degree of rationality of consumers, the length of their economic lifetime or the completeness of markets, there must be *some* link between asset returns and consumption, between prices and quantities. In the simplest case that we will explore in this paper—the complete markets, representative agent framework—this link has a name: the consumption function. The reason for the neglect of the consumption function and the almost exclusive focus on first-order conditions is obvious: it is difficult to solve for the consumption function in interesting problems. But technical difficulties are no valid reason for sticking with Euler equations and for neglecting the link between the two measures of risk represented by the covariance of asset returns with the wealth return or with consumption.

In this paper, we attempt to take these two remarks seriously. We develop an equilibrium capital asset pricing model based on Kreps-Porteus preferences—as exposed in Weil (), Epstein and Zin () and Giovannini and Weil ()—in which the marginal rate of substitution depends both on the rate of growth of consumption and on the rate of return on wealth. But, contrary to previous authors with the glaring exception of Campbell (), we make explicit (albeit through log-linear approximations) the links between consumption and wealth returns to characterize equilibrium excess returns.

Although our paper conforms to Campbell’s philosophy—we go beyond Euler equations by using the information contained in the consumption function—it takes a different perspective on the goals to be achieved. Campbell’s objective is to use the consumption function to eliminate consumption from his asset pricing expressions, or, as he puts it, to compute asset prices “without consumption data”. His rationale is that aggregate per capita consumption of non-durables and services i) is a poor measure for the consumption of market participants, and ii) is subject to measurement and time-aggregation errors. As a result, he derives expressions for excess returns that look like a generalized version of the market CAPM.

Our view, suggested at the outset, is that, from a data perspective, the difficulties involved with measuring the rate of return on wealth are as large as, if not larger than, those involved with measuring the consumption of market participants: the rate of return on total wealth is not simply mismeasured, it is not measured at all. Reversing Campbell’s method, we observe that consumer’s total wealth can be reconstructed *from consumption data alone* under the maintained assumption that the consumption data that we observe were generated by (Kreps-Porteus) utility maximizing agents. From these reconstructed total wealth data, we can compute an implied series of rates of return on total consumer wealth—which again is solely a function of consumption data. These reconstructed wealth returns can then be

In another paper, Campbell () attempts to circumvent the absence of data on the rate of return on human wealth by assuming that human wealth is constant fraction of total wealth, and that its return can be approximated by a linear function of labor income growth. Since human wealth is not the only component of wealth for which no data are available, those strong assumptions can hardly solve completely the data difficulties that motivate us.

used to calculate an (approximate) pricing kernel which, because it is in turn also solely a function of consumption data, yields a generalized consumption CAPM.

What we are doing can thus be thought as stripping the asset pricing methods of Lucas () or Mehra and Prescott () from their general equilibrium interpretation and from the “fruit tree” imagery: we take consumption as given, and we infer back from budget constraints and first-order conditions the wealth and the asset prices that support observed consumption as a utility-maximizing outcome. The (approximate) asset pricing kernel that we compute enables us to price any asset (including wealth) and determine its stochastic equilibrium returns solely as a function of its payoff and of observed consumption. This procedure allows us to price the stock market as a *subset* of wealth, and to compare empirically the implications of the model for stock market returns with the ones on the wealth portfolio.

The paper is organized as follows. We present the model, and the basics of our reconstruction of wealth from observed consumption data, in section . We then turn, in section , to the determination of asset prices in a world with a homoskedastic consumption process, postponing to section the analysis of equilibrium with heteroskedastic consumption. In section , we examine the implications of our model for the term structure of real interest rates. In section , we compute the predicted equilibrium returns on the wealth portfolio and compare them with stock market returns. In that section, we also review other empirical papers that make use of our approximate equilibrium asset pricing approach. The conclusion offers directions for future research.

2. THE MODEL

The economy consists of many identical infinitely-lived consumers. All wealth is assumed to be tradeable. Let W_t denote wealth at time t , and $R_{w,t}$ the rate of return on the “wealth portfolio” between dates $t - 1$ and t . Wealth can be accumulated in many forms, among which money, stocks, bonds, real estate, physical and human capital. The rate of return on wealth will be, in equilibrium, the rate of return on this exhaustive “market portfolio.”

A representative consumer faces the following budget constraint:

$$W_{t+1} = R_{w,t+1}(W_t - C_t). \quad (2.1)$$

In addition, our consumer’s initial wealth is given, and she faces a solvency constraint to rule out Ponzi games.

Following Epstein and Zin () and Weil (), we assume that consumers have Kreps-Porteus *generalized isoelastic preferences* (GIP) with a constant elasticity of substitution, $1/\rho$, and a constant (but in general unrelated) coefficient of relative risk aversion, γ , for timeless gambles. These preferences can be represented recursively as

$$V_t = \left\{ (1 - \beta)C_t^{1-\rho} + \beta(E_t V_{t+1})^{1/\theta} \right\}^\theta, \quad (2.2)$$

where $0 < \beta < 1$, V_t is the agent’s utility at time t , C_t denotes consumption, the operator E_t denotes mathematical expectation conditional on information

available at t , and the parameter

$$\theta = (1 - \gamma)/(1 - \rho)$$

measures the departure of the agents' preferences away from the time-additive isoelastic expected utility framework. Thus, when $\theta = 1$, the preferences in (.) reduce to the standard time-additive isoelastic expected utility representation.

. . **The Euler equation.** Epstein and Zin () have shown that for any asset with gross rate of return $R_{i,t+1}$ between dates t and $t + 1$ the following Euler equation must be satisfied:

$$E_t \left\{ \beta^\theta \left[\frac{C_{t+1}}{C_t} \right]^{-\rho\theta} R_{w,t+1}^{\theta-1} R_{i,t+1} \right\} = 1. \quad (.)$$

By log-linearizing (.) and subtracting from it the version of the Euler equation that holds for a safe one-period bond with gross rate of return $R_{f,t+1}$, we obtain the familiar expression for the (approximate) excess return on any asset:

$$E_t r_{i,t+1} - r_{f,t+1} = -\frac{\sigma_{ii,t}}{2} + \rho\theta\sigma_{ic,t} + (1 - \theta)\sigma_{iw,t}, \quad (.)$$

where lowercase letters denote the logarithm of their uppercase counterpart, and where $\sigma_{pq,t}$ denotes the conditional covariance at time t between random variables p_{t+1} and q_{t+1} .

This equation is often interpreted as implying that, for GIP preferences, excess returns are determined by a combination of the CCAPM and of the SCAPM. This is misleading, since consumption and the return on wealth (or $\sigma_{ic,t}$ and $\sigma_{iw,t}$) in general depend on each other, through the behavior of forward-looking, optimizing consumers who must satisfy the budget constraint (.).

. . **The (approximate) relation between consumption and asset returns.** Taking, as in Campbell (), a first-order Taylor expansion around the unconditional mean of the logarithm of the consumption-wealth ratio, we obtain the following approximate log-linear budget constraint:

$$r_{w,t+1} \approx x_{t+1} - a_{t+1} + \frac{1}{\delta}a_t - k \quad (.)$$

where $x_t = \log \frac{C_t}{C_{t-1}}$, $a_t = \log \frac{C_t}{W_t}$ and k and δ ($0 < \delta < 1$) are two easily computed linearization constants. Equation (.) implies that

$$S_{t+1} r_{w,t+1} = S_{t+1}(x_{t+1} - a_{t+1}). \quad (.)$$

See, for instance, Giovannini and Weil ().

See, for instance, Epstein and Zin () or Giovannini and Weil ().

Throughout, we assume stationarity. The assumption that log consumption-wealth ratio is stationary is validated in equilibrium.

The constant δ is equal to $1 - \exp[E(a)]$, and the constant k is equal to $\log(\delta/(1-\delta)) + (1/\delta) E(a)$. Campbell () examines the accuracy of this log-linear approximation, which depends on the variability of the consumption-wealth ratio in a model mathematically identical to ours but in which asset returns, instead of consumption, are treated as exogenous data. To the extent that consumption is less volatile than asset returns, our approximations are at least as good as those in Campbell ().

where, for any random variable q_{t+1} , the *surprise operator* S is defined as

$$S_{t+1} q_{t+1} \equiv E_{t+1} q_{t+1} - E_t q_{t+1} = q_{t+1} - E_t q_{t+1}.$$

An implication of (.) is that, if the budget constraint is satisfied, the conditional covariance of any asset return with the rate of return on wealth is just the difference between, on the one hand, the conditional covariance of this asset's return with consumption and, on the other hand, the conditional covariance of this asset's return with the (log) propensity to consume. Namely,

$$\sigma_{iw,t} = \sigma_{ic,t} - \sigma_{ia,t}. \quad (.)$$

While this equation is not operational (we have not yet said anything about $\sigma_{ia,t}$), it has the merit of pointing out that the covariance between individual returns and the return on the wealth portfolio is endogenous, through its dependence on the covariance $\sigma_{ia,t}$ between individual returns and the still to be computed endogenous propensity to consume.

. . **Eliminating the rate of return on wealth.** Since our goal in this paper is to derive asset pricing expressions that do not involve the unobservable rate of return on the wealth portfolio, we can use equation (.) to eliminate the terms involving the rate of return on wealth from the excess returns expression (.):

$$E_t r_{i,t+1} - r_{f,t+1} = -\frac{\sigma_{ii,t}}{2} + \gamma \sigma_{ic,t} + (\theta - 1) \sigma_{ia,t}. \quad (.)$$

Equation (.) highlights two important special cases that are explored systematically in Giovannini and Weil (.):

- *in the expected utility case* ($\theta = 1$), equation (.) is the excess return equation characteristic of the CCAPM.
- *when the consumption-wealth ratio is constant* ($a_t = a$ for all t), equation (.) implies that asset returns must also conform to the CCAPM, but that model should then be equivalent to the SCAPM, since consumption growth and the rate of return on wealth are then perfectly correlated.

. . **The consumption-wealth ratio.** The expression for excess returns in (.) is still non-operational, as the extra-term introduced by GIP preferences, $\sigma_{ia,t}$, depends on the propensity to consume that we have not yet calculated. We now take up the task of characterizing the optimal consumption-wealth ratio.

From the version of the Euler equation (.) that holds for the wealth return, it follows that, when $\theta \neq 0$,

$$E_t r_{w,t+1} = -\log \beta + \rho E_t x_{t+1} - \frac{\theta}{2} \text{Var}_t(r_{w,t+1} - \rho x_{t+1}) \quad (.)$$

Now, using (.), and solving for a_t in (.) we find

$$a_t \approx \delta \left(k - \log \beta + E_t[a_{t+1} - (1 - \rho)x_{t+1}] - \frac{\theta}{2} \text{Var}_t[a_{t+1} - (1 - \rho)x_{t+1}] \right). \quad (.)$$

It is at this point that we depart from Campbell (.).
See, for instance, the budget constraint (.).

Consistent with our approach that seeks to express all variables in terms of consumption, we interpret (.) as a difference equation in the a 's driven by the x 's. Under the transversality condition $\lim_{s \rightarrow \infty} \delta^s a_{t+s} = 0$, (.) implies that

$$a_t = \frac{\delta(k - \log \beta)}{1 - \delta} - (1 - \rho) E_t \sum_{j=1}^{\infty} \delta^j x_{t+j} - \frac{\theta}{2} E_t \sum_{j=1}^{\infty} \delta^j \text{Var}_{t+j-1} z_{t+j}, \quad (.)$$

where

$$z_t \equiv a_t - (1 - \rho)x_t. \quad (.)$$

Two remarks are in order. First, (.) still does not provide the solution for the consumption wealth ratio a_t as a function of consumption and preferences, since conditional first moments of future conditional second moments appear on the right-hand side. But as we shall see below, (.) does provide a clue as to the functional form of the solution. Second, uniqueness of the solution (when the solution exists) is guaranteed by the fact that the transversality condition, the Euler equation and the budget constraint (all of which are imbedded in (.)) are jointly necessary and sufficient for a unique solution to the optimal consumption problem we are approximating.

We are now ready to turn to the determination of equilibrium returns. We examine two cases in the next two sections. First, a case in which log consumption growth is conditionally homoskedastic. Second, a case in which consumption is conditionally heteroskedastic and follows an AR() process with GARCH(,) disturbances.

2. E R : H C

Suppose the log consumption growth and the conditional mean of future log consumption growth are jointly conditionally homoskedastic, so that the conditional variance of consumption growth, its conditional covariance with future expected consumption growth and the conditional variance of future expected consumption growth are constant over time.

2.1. The propensity to consume. When conditional second-order moments are constant, it is straightforward to check that the solution to (.) is simply

$$a_t = g - (1 - \rho) E_t \sum_{j=1}^{\infty} \delta^j x_{t+j} \quad (.)$$

where the constant g is given by

$$g \equiv \frac{\delta}{1 - \delta} \left[k - \log \beta - \frac{\theta(1 - \rho)^2}{2} (\sigma_{cc} + \sigma_{hh} + 2\sigma_{ch}) \right],$$

This transversality condition, which is also used by is also used by Campbell (), is part of the necessary conditions for optimality. It rules out bubbles in this Ramsey-type economy.

and where σ_{hh} and σ_{ch} respectively denote, under the notation

$$h_{t+1} \equiv E_{t+1} \sum_{j=1}^{\infty} \delta^j x_{t+j+1},$$

the conditional variance of expected discounted future consumption growth, and the conditional covariance between consumption growth and expected discounted future consumption growth.

The interpretation of (.) is the following: *Given current wealth*, high expected future consumption can only stem from high expected future returns on wealth. Thus, if the elasticity of intertemporal substitution is large ($\rho < 1$)—i.e., if substitution effects are stronger than income effects—our consumer reacts negatively to high expected future returns by consuming less, so that the consumption-wealth ratio declines. If, on the other hand, the elasticity of intertemporal substitution is small ($\rho > 1$), high expected future consumption has a negative impact on the propensity to consume. Equation (.) is the mirror image of these effects *given consumption*.

Note that, as a result of (.), homoskedasticity of consumption growth and future expected consumption growth implies homoskedasticity of the consumption wealth ratio.

. . The rate of return on wealth. From (.) and (.), we can reconstruct the approximate equilibrium rate of return on the (unobservable) wealth portfolio:

$$r_{w,t+1} = u + \rho x_{t+1} + (1 - \rho) S_{t+1} \sum_{j=0}^{\infty} \delta^j x_{t+j+1}, \quad (.)$$

where

$$u = -\log \beta - \frac{\theta(1 - \rho)^2}{2} (\sigma_{cc} + \sigma_{hh} + 2\sigma_{ch}). \quad (.)$$

This equation (.) enables us to compute, date by date and state by state, the return on the wealth portfolio from observable consumption data alone. This equilibrium rate of return on wealth has to be understood as the return on wealth which supports, under the assumption that the model is true, the consumption process as an *equilibrium* consumption path. In other terms, equation (.) allows us to reconstruct the unobserved return on wealth from observed consumption data.

An implication of equation (.) is that:

$$E_t r_{w,t+1} = u + \rho E_t x_{t+1}. \quad (.)$$

As a consequence, when expected future consumption raises, consumers require higher expected returns in order to offset their distaste for intertemporal substitution. Moreover, in this homoskedastic world, the magnitude of the increase in conditional expected return on wealth that is required to compensate an increase in the conditional expected rate of growth of consumption is exactly the inverse of the elasticity of intertemporal substitution—i.e., ρ .

. . **The approximate pricing kernel.** The expression (.) for the implied rate of return of the wealth portfolio enables us to compute the (approximate) equilibrium pricing kernel for this economy as a function of the consumption process. From the Euler equation (.), it follows that the log marginal rate of substitution between periods t and $t + 1$ is

$$m_{t+1} = \theta \log \beta - \rho \theta x_{t+1} + (\theta - 1) r_{w,t+1}.$$

Substituting (.) into this expression and rearranging, we find that

$$m_{t+1} = v - \rho x_{t+1} + (\rho - \gamma) S_{t+1} \sum_{j=0}^{\infty} \delta^j x_{t+j+1},$$

where $v = \theta \ln \beta + (\theta - 1)u$. In the standard time and state-additive case ($\gamma = \rho$) and/or in an i.i.d. world ($S_{t+1} \sum_{j=1}^{\infty} \delta^j x_{t+j+1} = 0$), the (log) pricing kernel is, up to a constant, a linear function of the (log) consumption growth rate. In all other cases, it depends in addition on the news received at time $t + 1$ about consumption growth rates in periods $t + 2$ and beyond.

. . **Excess returns.** It follows from equation (.) that surprises in the propensity to consume are given by

$$S_{t+1} a_{t+1} = -(1 - \rho) S_{t+1} \sum_{j=1}^{\infty} \delta^j x_{t+j+1}, \quad (.)$$

so that the conditional covariance between the return on any asset and the marginal propensity to consume is

$$\sigma_{ia} = -(1 - \rho) \sigma_{ih} \quad (.)$$

where σ_{ih} denotes the conditional covariance between the return on asset i and expectations of future (discounted) consumption growth.

Therefore, substituting (.) into (.), the equilibrium excess return on any asset satisfies

$$E_t r_{i,t+1} - r_{f,t+1} = -\frac{\sigma_{ii}}{2} + \gamma \sigma_{ic} + (\gamma - \rho) \sigma_{ih}. \quad (.)$$

According to this expression, the excess return on any asset depends on its own variance (a Jensen's inequality term), on its conditional covariance with contemporaneous consumption, and on its conditional covariance with future consumption. To understand (.), it is best to think of time as consisting of three dates: today, tomorrow, and the day after tomorrow (or future), and to examine separately the three terms $\gamma \sigma_{ic}$, $\gamma \sigma_{ih}$, and $-\rho \sigma_{ih}$ that govern excess returns:

- An asset with $\sigma_{ic} > 0$ is an asset whose return between today and tomorrow tends to be high (low) when consumption tomorrow is high (low). Holding such an asset in one's portfolio makes it difficult to smooth consumption over states of nature. Therefore, risk averse investors require a premium over the riskless return to hold this asset. This premium is larger the larger the consumers' aversion to substitution over states of nature, i.e., the larger their coefficient of relative risk aversion γ . The presence of the term $\gamma \sigma_{ic}$

on the right hand-side of (.) thus reflects our consumer's *aversion to substitution over states of nature*.

- An asset with $\sigma_{ih} > 0$ is an asset whose return between today and tomorrow tends to be high (low) when there are good (bad) news about consumption the day after tomorrow. Such an asset is not attractive, as it provides, say, more wealth tomorrow when good news about future consumption make it less desirable to be able to save for precautionary motives. As a result, our consumers require a premium to hold this asset, and the term $\gamma\sigma_{ih}$ reflects the desire of our consumers' *precautionary saving motive*.
- However, an asset with $\sigma_{ih} > 0$ is desirable for consumers who dislike fluctuations of consumption across dates, as holding such an asset smoothes the intertemporal consumption profile. Therefore, the more consumers are averse to intertemporal substitution (the larger ρ), the more willing they are to hold an asset with $\sigma_{ih} > 0$, and the smaller the excess return required in equilibrium to induce consumers to hold this asset. This explains the presence of the $-\rho\sigma_{ih}$ term, which reflects our consumers' *aversion to intertemporal substitution*.

Two special cases of (.) are noteworthy:

- When $\gamma = \rho$, the precautionary saving and intertemporal substitution effects cancel out. It is thus an unfortunate feature of standard isoelastic preferences that they hide two fundamental determinants of equilibrium excess returns.
- When $\gamma = 0$ and $\rho > 0$, i.e., when consumers have no desire to smooth consumption over states and do not engage in precautionary saving, excess returns will generally be non-zero. There is nothing pathological about this: Drèze and Modigliani () have taught us about the temporal dimension of risk aversion. A zero aversion to atemporal risk ($\gamma = 0$) does not imply a zero risk premium as long as one is not indifferent to intertemporal substitution ($\rho > 0$). It is, therefore, another unfortunate feature of standard isoelastic preferences that they associate zero risk aversion to atemporal gambles with zero aversion to intertemporal substitution and thus to zero risk premia: this is simply not a general result.

Finally, notice that, unlike the excess return equation derived by Campbell (), equation (.) does assign a role to the intertemporal elasticity of substitution in the determination of equilibrium excess returns. This is because the covariances that appear in (.) are covariances with consumption, in contrast with the covariances in Campbell () which are covariances with the market return. What happens is that expressing excess returns as a function of covariances with market returns “hides” the ρ coefficient into the covariance terms, since an implication of equation (.) is that, for $j > 1$,

$$S_{t+1} r_{w,t+j} = \rho S_{t+1} x_{t+j}. \quad (.)$$

Our consumers, because they have risk preferences with constant relative risk aversion, do save for precautionary motives and have decreasing absolute prudence.

Both the second and third derivatives of the risk utility function are zero when $\gamma = 0$.

Thus, one must exercise care when characterizing the role of aversion to intertemporal substitution in the determination of excess returns. From the equilibrium perspective that we adopt in this paper, it is only natural that excess returns depend on the elasticity of intertemporal substitution, as investors care about the temporal properties of asset payoffs (do they help smooth consumption over *time*?).

. . The excess return on wealth. We now turn to the computation of the equilibrium excess return on wealth. From (.),

$$S_{t+1} r_{w,t+1} = S_{t+1} x_{t+1} + (1 - \rho) S_{t+1} \sum_{j=1}^{\infty} \delta^j x_{t+j+1}. \quad (.)$$

As a consequence,

$$\sigma_{wc} = \sigma_{cc} + (1 - \rho)\sigma_{ch} \quad (.)$$

$$\sigma_{wh} = \sigma_{ch} + (1 - \rho)\sigma_{hh} \quad (.)$$

Substituting into (.), we obtain the following formula for the equilibrium excess return on wealth (up to a Jensen's inequality term brought for clarity to the left-hand side):

$$\begin{aligned} E_{t+1} r_{w,t+1} - r_{f,t+1} + \sigma_{ww}/2 \\ = \gamma[\sigma_{cc} + (1 - \rho)\sigma_{ch}] + (\gamma - \rho)[\sigma_{ch} + (1 - \rho)\sigma_{hh}]. \end{aligned} \quad (.)$$

When returns are i.i.d., all the terms involving h are zero, and the rate of return on wealth is equal to $\gamma\sigma_{cc}$ regardless of whether the expected utility restriction is satisfied: the excess return on the wealth portfolio is then determined solely by risk aversion and the variance of consumption growth. This is not surprising, as time—and thus the coefficient aversion to intertemporal substitution, ρ —is essentially irrelevant in an i.i.d. world. As soon as we depart from the i.i.d. world, however, the “wealth premium” depends on both aversion to risk and aversion to intertemporal substitution.

Note that the rate of return on wealth is simply, from (.), the rate of return on a claim to aggregate consumption—a concept that has, in general, little to do empirically with the rate of return on the equity traded in the stock market. We will come back to this issue in section .

. . Prices. One should note that the expression in (.) does not provide us with a formula to compute the equilibrium excess return on an asset as a function of its payoff structure, the consumption process and preferences. The reason is, of course, that the endogenous return r_i appears in the conditional second order moments on the right-hand side of (.). To find such a *bona fide* asset pricing

Similar results are noted in Weil ().

The two returns are however identical *by construction* in the Lucas () or Mehra and Prescott () models.

formula, we first need to consider how the return on an asset depends on its price and the dividends (payoffs) it distributes.

Let $p_{i,t}$ denote the log of the (*cum* dividend) price-dividend ratio of asset i at time t , and $d_{i,t}$ the log rate of growth of the dividends paid off by asset i between dates t and $t + 1$. Then, by definition, the log return on asset i satisfies the identity:

$$r_{i,t+1} \equiv d_{i,t+1} + p_{i,t+1} - \log(e^{p_{i,t}} - 1). \quad (.)$$

Following Campbell and Shiller (), we assume that the log dividend growth process is stationary and use a Taylor expansion similar to the one applied above to the budget constraint to find that

$$r_{i,t+1} \approx d_{i,t+1} + p_{i,t+1} - \frac{1}{\delta_i} p_{i,t} - k_i, \quad (.)$$

where k_i and δ_i ($0 < \delta_i < 1$) are two linearization constants. Since wealth is simply an asset that distributes a dividend equal to per capita aggregate consumption, the approximate budget constraint () is but a special case of () with $d_{i,t} = x_t$, $p_{i,t} = -a_t$, $\delta_i = \delta$, and $k_i = k$.

Now, it follows from () that, because of homoskedasticity, the expected rate of return on asset i differs from the expected rate of return on wealth only by a constant, call it π_{iw} :

$$\pi_{iw} = E_t r_{i,t+1} - E_t r_{w,t+1}. \quad (.)$$

Therefore, applying conditional expectations to both sides of (), substituting () into the resulting expression, and iterating () forward using the property that bubbles are infeasible in this economy, we find that

$$p_{i,t} = -(k_i + \pi_{iw}) \frac{\delta_i}{1 - \delta_i} + E_t \sum_{s=1}^{\infty} \delta_i^s [d_{i,t+s} - r_{w,t+s}]. \quad (.)$$

The only term on the right-hand side of this expression that we do not yet know how to compute from consumption data alone is π_{iw} . Now notice that we can rewrite

$$\pi_{iw} = E_t r_{i,t+1} - E_t r_{w,t+1} = [E_t r_{i,t+1} - r_{f,t+1}] - [E_t r_{w,t+1} - r_{f,t+1}].$$

We have already computed the equilibrium excess return of the wealth portfolio in (), so that the only task left is to characterize the excess return on individual assets.

Notice that () implies

$$S_{t+1} p_{i,t+1} = S_{t+1} \sum_{s=1}^{\infty} \delta_i^s [d_{i,t+s+1} - r_{w,t+s+1}], \quad (.)$$

so that, using (),

$$S_{t+1} p_{i,t+1} = S_{t+1} \sum_{s=1}^{\infty} \delta_i^s [d_{i,t+s+1} - \rho x_{w,t+s+1}]. \quad (.)$$

That is, the log of the cum dividend price minus the log of the dividend.

Now, from (.),

$$S_{t+1} r_{i,t+1} = S_{t+1} d_{i,t+1} + S_{t+1} p_{i,t+1}.$$

Therefore, from (.), we find that

$$S_{t+1} r_{i,t+1} = S_{t+1} d_{i,t+1} + S_{t+1} \sum_{s=1}^{\infty} \delta_i^s d_{i,t+s+1} - \rho S_{t+1} \sum_{s=1}^{\infty} \delta_i^s x_{t+s+1}. \quad (.)$$

The interpretation of (.) is straightforward. Good news about the rate of return on asset i can come from good news about tomorrow's dividends or future dividends (the first two terms on the right-hand side). Or they can come from news that future consumption growth will be low (the third term on the right-hand side), since, by (.), bad news about future consumption growth translate, in equilibrium, into news that future returns will be low, and, therefore, into news that the present discounted value of future dividends is high. The more averse the consumers are to intertemporal substitution (the larger ρ), the more sensitive equilibrium returns are to changes in consumption growth, and the more bad news about future consumption means good news for current returns.

Equation (.) immediately implies that

$$\sigma_{ic} = \sigma_{d_ic} + \sigma_{f_ic} - \rho \sigma_{h_ic}, \quad (.)$$

$$\sigma_{ih} = \sigma_{d_ih} + \sigma_{f_ih} - \rho \sigma_{h_ih}, \quad (.)$$

with the notation

$$f_{i,t+1} \equiv E_{t+1} \sum_{j=1}^{\infty} \delta_i^j d_{i,t+j+1} \quad \text{and} \quad h_{i,t+1} \equiv E_{t+1} \sum_{j=1}^{\infty} \delta_i^j x_{t+j+1}.$$

Thus, for instance, σ_{h_ic} measures the conditional covariance between expected discounted future dividend growth of asset i and tomorrow's consumption, while σ_{h_ih} measures the conditional covariance between two differently discounted expectations of future consumption.

Substituting (.) and (.) into (.), and collecting terms, we find that the equilibrium excess return on any asset i is given by

$$\begin{aligned} E_t r_{i,t+1} - r_{f,t+1} = & - \sigma_{ii}^2/2 \\ & + \gamma \sigma_{d_ic} + (\gamma - \rho) \sigma_{d_ih} \\ & + \gamma \sigma_{f_ic} + (\gamma - \rho) \sigma_{f_ih} \\ & - \rho [\gamma \sigma_{h_ic} + (\gamma - \rho) \sigma_{h_ih}]. \end{aligned} \quad (.)$$

Equation (.) computes the equilibrium excess return on asset i solely as a function of the moments of this asset's dividend growth process and of the consumption growth process. The interpretation of (.) runs, of course, very much along the lines of the interpretation of (.). We showed in (.) that there are three behavioral determinants of excess returns: aversion to risk, prudence, and aversion to intertemporal substitution. The excess return equation (.) simply shows that each of these behavioral determinants applies to the each of the events, described in (.), associated with good news about the return on asset i : news

that tomorrow's dividends will be high, that future dividends will be high, or that future consumption growth will be low.

To complete the computation of equilibrium prices, all that remains to be done is to subtract from (.) the equilibrium excess return on wealth computed in (.). This will yield the constant difference, π_{iw} , between the rate of return on asset i and the rate of return on wealth. Using the expression in (.) for the rate of return on wealth, and substituting the just computed π_{iw} into (.), would yield the (approximate) equilibrium price of any asset i as a function of consumption and dividend data alone.

$$. E \quad R \quad : H \quad C$$

Obtaining a closed-form solution for equation (.) when consumption follows a general heteroskedastic process is not feasible (it requires computing conditional moments of conditional moments of conditional moments etc.). We can however extend the results of the previous section to a relatively common heteroskedastic case by assuming that log consumption growth follows a Gaussian GARCH process. In particular, we will assume here the simple AR()-GARCH(,) specification:

$$x_{t+1} = a + bx_t + u_{t+1} \quad (.)$$

$$u_{t+1} \sim |_t \mathcal{N}(0, \sigma_{cc,t}) \quad (.)$$

$$\sigma_{cc,t} = \alpha_0 + \alpha_1 u_t^2 + \alpha_2 \sigma_{cc,t-1} \quad (.)$$

We will use three properties of GARCH processes that are established in Restoy (). If two random variables have a joint normal conditional distribution whose second order moments follow GARCH processes analogous to (.), then:

Property 1: Today's conditional expectation of products of powers of tomorrow's conditional second order moments is a polynomial in today's conditional second order moments.

Property 2: Today's conditional covariance between products of powers of tomorrow's conditional second order moments is a polynomial in today's conditional second order moments.

Property 3: Today's conditional covariance between one of these random variables tomorrow and product of powers of tomorrow's conditional second order moments is zero.

. . **The consumption-wealth ratio.** Properties and immediately imply that the solution to (.) (i.e., the equilibrium consumption wealth ratio) can be written as

$$a_t = n - (1 - \rho) \frac{\delta b}{1 - \delta b} x_t + \sum_{j=1}^{\infty} \zeta_j \sigma_{cc,t}^j, \quad (.)$$

For the sake of brevity, we do not present the resulting formula.

Recent examples of using GARCH specifications to model the conditional variance of consumption are Piazzesi (), Lettau and Ludvigson (), and Du ee ().

The straightforward proofs can be found there in Lemmas , , and .

where the constant n and the ζ_j coefficients—which are, as we shall see below, uninformative and irrelevant for excess returns—can be computed as in Restoy (2010).

To understand this equation, it is best to compare it with (2.10). The term in x_t on the right-hand side of (2.10) represents the expected present discounted value of future consumption, which is just a linear function of current consumption growth because of the AR(1) process followed by consumption growth. The polynomial in the current conditional variance of consumption is present by virtue of Properties 1 and 2, which guarantee that the last term in (2.10) can be expressed in the form, given in (2.11), of a weighted sum of powers of the current conditional variance of consumption.

2.2. **Excess returns.** Property 1 implies

$$\text{Cov}_t \left(r_{i,t+1}, \sum_{j=1}^{\infty} \zeta_j \sigma_{cc,t+1}^j \right) = 0.$$

As a consequence, from (2.10) and (2.12),

$$\sigma_{ia,t} = \frac{\delta b}{1 - \delta b} (\rho - 1) \sigma_{ic,t}. \quad (2.13)$$

This is an important result because it embodies the fundamental insight that, for our AR(1)–GARCH(1,1) process, returns are only able to predict future conditional *means* of consumption growth but carry no information about the future conditional *variances*. Therefore, the ζ_j parameters are irrelevant when it comes to computing excess returns, and the parameters of GARCH process do not matter for excess returns! Indeed, substituting (2.13) into equation (2.10), one obtains

$$E_t r_{i,t+1} - r_{f,t+1} = -\frac{\sigma_{ii}}{2} + \left[\gamma + (\gamma - \rho) \frac{\delta b}{1 - \delta b} \right] \sigma_{ic,t}. \quad (2.14)$$

Because of Properties 1 to 3, this expression is almost identical formally to the one we would have obtained, in (2.10), for an AR(1) process with homoskedastic errors. Because of the autoregressive nature of consumption growth, the only conditional moment that matters for excess returns is the current conditional covariance between asset returns and consumption. But the one crucial distinction is that excess returns now vary over time, reflecting the time variation of the conditional variance of log consumption growth.

While one might be tempted to conclude from (2.14) that this model is observationally equivalent to a standard CCAPM model with coefficient of relative risk aversion (or inverse of the elasticity of intertemporal substitution)

$$\gamma' = \gamma + (\gamma - \rho) \delta b / (1 - \delta b),$$

this would be mistaken. If γ is small relative to ρ and consumption growth is highly persistent, the implied γ' might well be negative, and the excess return on an asset might be negative when the conditional covariance between that asset's return and consumption is positive.

A particular case is when the consumption growth rate is not persistent ($b = 0$), but exhibits conditional heteroskedasticity of the GARCH form. From (.), that assumption implies that the CCAPM's excess returns expression holds. Similarly, equations (.), (.) and Property imply that the SCAPM also holds. This result shows how i.i.d. consumption growth (as in Kocherlakota ()) is a sufficient but not necessary distributional assumption to get observational equivalence between SCAPM, CCAPM and the excess return expression associated to the model with GIP preferences. Notice however that, even in this case, it is not true that elasticity of intertemporal substitution is irrelevant to determine asset prices as long as it affects the equilibrium rate of return on wealth.

$$\begin{array}{ccccc} & T & & R & A \\ T & S & & R & I & R \end{array}$$

The previous sections have highlighted in several instances the fact that risk neutrality towards timeless gambles does not imply that excess premia should be zero for all assets regardless of their maturity. As we emphasized above, the latter presumption is valid only in the time- and state-additive expected utility case—for, in that case, neutrality towards timeless risks coincides with indifference to the date at which one consumes, and thus to the irrelevance of the time dimension of risk. However, this coincidental result does not carry over to more general setups, and there is no blanket presumption that equilibrium risk premia should be zero at all maturities when consumers are neutral towards timeless risks—which confirms in equilibrium the partial equilibrium analysis of Drèze and Modigliani ().

To highlight the role of temporal risk aversion, we now return to the homoskedastic case and characterize the equilibrium term structure of real bond returns under the assumption that the log consumption growth process follows an homoskedastic, AR() process:

$$x_{t+1} = a + bx_t + \epsilon_{c,t+1}, \quad (.)$$

$$\epsilon_{c,t+1} \sim |_t \mathcal{N}(0, \sigma_{cc}). \quad (.)$$

We consider pure discount bonds maturing $j \geq 1$ periods from now, i.e., riskfree claims that promise to pay one unit of the consumption good in every state of nature j periods from now. Let $R_t(j)$ denote the gross one-period return at time t on a bond of maturity j . It is straightforward to show that $R_t(j)$ must satisfy the following Euler equation:

$$E_t \left\{ \beta^{j\theta} \prod_{k=1}^j X_{t+k}^{-\rho\theta} \prod_{k=1}^j R_{w,t+k}^{\theta-1} \right\} [R_t(j)]^j = 1. \quad (.)$$

Computations are more tedious, but the results not more instructive, in the heteroskedastic case.

The one-period rate of return at t on a bond maturing at $t + 1$, $R_t(1)$, is simply what we called earlier $R_{f,t+1}$.

Similarly, the return on a j -period *rolling over short* strategy must satisfy

$$E_t \left\{ \beta^{j\theta} \prod_{k=1}^j X_{t+k}^{-\rho\theta} \prod_{k=1}^j R_{w,t+k}^{\theta-1} \prod_{k=0}^{j-1} R_{t+k}(1) \right\} = 1. \quad (.)$$

In the appendix we show that, under the same joint lognormality assumption we used above, the Euler equation corresponding to the j -period bond can be written as.

$$r_t(j) = -\log \beta + \rho S(a, b, j) + \rho T(b, j) x_t + \frac{1}{2} \left\{ \frac{(\rho - \gamma)(1 - \gamma)}{(1 - \delta b)^2} - A(b, j) \left[\rho + \frac{\gamma - \rho}{1 - \delta b} \right]^2 \right\} \sigma_{cc}, \quad (.)$$

where

$$\begin{aligned} S(a, b, j) &\equiv \frac{1}{j} \frac{a}{1 - b} \left[j - b \frac{1 - b^j}{1 - b} \right], \\ T(b, j) &\equiv \frac{b}{j} \frac{1 - b^j}{1 - b}, \\ A(b, j) &\equiv \frac{1}{j} \sum_{k=1}^j \frac{1 - b^{2k}}{1 - b^2} \left[1 + 2b \frac{1 - b^{j-k}}{1 - b} \right]. \end{aligned}$$

Equation (.) allows us to draw (approximate) yield curves for pure discount bonds. In this homoskedastic world, those yield curves would be flat if consumption is *i.i.d.* ($b = 0$) and/or agents have an infinite elasticity of intertemporal substitution ($\rho = 0$).

In the appendix we also show that the rolling over short strategy yields a return which can be written as the return on a j -period bond plus a term premium. This term premium has the form:

$$\begin{aligned} TP(j) &= \rho b \left[-\frac{1}{2} \rho b + \rho + \frac{\gamma - \rho}{1 - \delta b} \right] A(b, j - 1)(j - 1) \\ &\quad + \rho b \left[\rho + \frac{\gamma - \rho}{1 - \delta b} \right] \sum_{k=1}^{j-1} b^{j-k} \frac{1 - b^{2k}}{1 - b^2}. \quad (.) \end{aligned}$$

The term premium is a complex function of the persistence parameter b and the preference parameters γ and ρ . Under the standard time-additive expected utility preferences, the term premium is zero if agents are risk neutral—because zero risk aversion is then associated with zero aversion to intertemporal substitution ($\gamma = \rho = 0$). In general, however, a zero coefficient of relative risk aversion for timeless gambles does *not* imply a zero term premium. By contrast, if agents have an infinite elasticity of intertemporal substitution ($\rho = 0$), the term premium is zero in equilibrium regardless of the value of the coefficient γ : when consumers do not care when they consume, the rate of return on a long bond and on the corresponding rolling over short strategy must be identical. Finally, note that the term premium is, of course, always zero if consumption is *i.i.d.*

. A

The previous sections suggest that our approximate equilibrium asset pricing approach section is well-suited to derive simple stochastic relations between security returns and fundamentals. We now show how to apply it empirically to the computation of equilibrium stock market returns, before turning to the discussion of related work.

. . Equilibrium stock market returns. If, as is often assumed, the stock market reduced to a claim on aggregate future consumption and could thus be identified as the wealth portfolio, equation (.) would enable us to approximate equilibrium equity returns r_e by the expression

$$r_{e,t+1} = r_{w,t+1} = u + \rho x_{t+1} + (1 - \rho) S_{t+1} \sum_{j=0}^{\infty} \delta^j x_{t+j+1}, \quad (.)$$

In a simple Lucas-type general equilibrium setting, x would represent (the growth rate of) consumption or output.

In the more general, and more realistic, case in which other components of aggregate wealth — such as human capital — force us to distinguish conceptually equity from the wealth portfolio, section . tells us how to compute (approximately) the rate of return on the stock market as a claim on future aggregate dividends. By combining equations (.), (.) and (.), we find:

$$r_{e,t+1} = u + \pi_{ew} + S_{t+1} \sum_{s=0}^{\infty} \delta_i^s d_{e,t+s+1} + \rho x_{t+1} - \rho S_{t+1} \sum_{s=0}^{\infty} \delta_i^s x_{t+s+1}, \quad (.)$$

where π_{ew} is a constant defined as in equation (.), and d_e denotes the growth rate of the aggregate dividends corresponding to the securities included in the stock-market index. If $d_e = x$, i.e., if the dividends we are pricing are aggregate consumption, $\pi_{ew} = 0$ and equation (.) reduces to (.).

In order to get an idea of how restrictive it is in practice to assume that the stock market is an accurate proxy of the economy's wealth, we can thus compare the empirical predictions of expressions (.) and (.) and confront them with observed stock market returns.

However, to run this empirical experiment we must first generate *surprises* for future expected discounted values of x and d . To that effect, we follow Campbell () and specify a VAR model for a set of variables of interest. Define Z as a -component vector composed of the real stock return, the growth rate of consumption (or output), the growth rate of dividends, and a (detrended) short term real interest rates. The vector Z is assumed to follow the first-order VAR process:

$$Z_{t+1} = \alpha + AZ_t + \omega_{t+1}. \quad (.)$$

Now observe that the vector of surprises to future discounted values of Z is given by the vector ξ , with

$$\xi_{t+1} \equiv S_{t+1} \sum_{j=0}^{\infty} \delta^j Z_{t+1+j} = \sum_{j=0}^{\infty} \delta^j A^j \omega_{t+1} = (I - \delta A)^{-1} \omega_{t+1}. \quad (.)$$

The second and third components of vector ξ correspond to *surprises* on future expected discounted values of x and d , respectively. After those surprise terms are generated by estimating of the VAR model, we can make use of the equilibrium expressions (.) and (.) to derive, for each value of ρ , the theoretical equity returns (up to a constant), and to compare them with actual data. In order to consider partial as well as general equilibrium interpretations of our model, we calculate x in expression (.) using either consumption or output data (since consumption ought to equal output in the Lucas general equilibrium model). Hence we obtain three series of equilibrium returns for each value of the elasticity of intertemporal substitution:

- *Consumption model*: a series pricing a claim to future consumption which is derived from expression (.) with x taken as consumption, as theory suggests it should be.
- *Output model*: a series pricing a claim to future output which is derived from expression (.) with x being measured as output.
- *Dividend model*: a series pricing a claim to future aggregate dividends d which is derived from expression (.), with x being consumption.

To keep the VAR structure simple, we use annual data corresponding to the second half of the previous century. This frequency seems also more appropriate since it could be expected that a pure equilibrium analysis would normally have little power to explain very short-run fluctuations in asset prices.

Table presents the correlation of each of the three series of generated returns with observed US stock returns for a wide range of parameter values. Notice that given the large measurement problems that the procedure involves — particularly regarding the computation of the expectational terms in the equilibrium expressions — we should not expect very high correlations even if the model were a good description of reality. Still, some of the results are striking. First, the performance of both the consumption and the output models is extremely poor — even accounting for measurement problems — for all values of ρ above . , as the correlation between theoretical and observed returns is virtually zero or negative in those cases. This is important because available empirical exercises tend to yield estimates of ρ , the inverse of the elasticity of substitution, somewhere between and (see e.g. Epstein and Zin,). Second, although returns generated by the dividend model do not show a strong association with observed returns, the correlation is always above the ones obtained with the other two models, only gets negative for values of ρ of or more and is around . when ρ is in the neighborhood of .

US data from to . Aggregate (value-weighted) stock market returns and dividends are obtained from CRSP tapes; consumption (non-durable and services), output (GDP), interest rates and inflation (calculated with consumption deflators) are obtained from OECD's main economic indicators.

An empirical explanation of the relative performance of the models considered is relatively straightforward. As it has been often reported (see e.g. Estrella and Mishkin, 1998), equity prices contain some leading information on future economic activity and firms' profits. This translates into a positive covariance of current returns with news on future consumption, output and dividends. By looking at expression (2.10), good news on future consumption (or output) has two opposite effects on the return on the equilibrium portfolio. First, it increases the expected future payoff. Second, it increases the expected discount factor to be applied to any future payoff. If the coefficient ρ is sufficiently large, the latter predominates thereby generating a negative covariance between current equilibrium returns and news on future consumption that translates into a negative correlation between theoretical and observed returns. In other words, the exercise shows that for empirically plausible values of ρ , identifying the stock market with the economy's wealth would imply a counterfactual equilibrium relation between equity prices and future economic activity. If the stock market is not deemed a claim on future consumption or output, as in (2.11), the payoff and the discounted factor effects are disconnected with each other. Still, even if returns are positively correlated with changes in expectations about future dividends, the discount factor effect implies that the correlation between observed and equilibrium returns decreases when ρ increases. But if, as is the case in the sample, the association between returns and future expected dividends is sufficiently high, it is still possible to find a reasonable correlation between model-generated returns and actual data. In any case, the evidence gathered here points to the difficulty of reconciling empirically the standard intertemporal asset pricing models with the common but restrictive assumption that the stock market can proxy for the aggregate wealth portfolio.

2.2. Related empirical work. The approximate equilibrium methodology has already been used in several papers to carry out various empirical exercises.

For example, Rodríguez et al. (2000) use expression (2.10) and the corresponding VAR model to test whether macroeconomic fundamentals can explain both the volatility and predictability of asset returns in eight OECD countries. Although they generally obtain mixed results, they find favorable evidence for the model in the case of the US when annual data are employed.

Dumas et al. (2002), building on this paper, consider an open economy general equilibrium model in which a country's aggregate dividends are assumed equal to this country's output. They test whether the distribution of national stock market returns can be explained by a single international discount factor model — such as the last term in the righthand side of equation (2.10) — depending on world output. Although Dumas et al. (2002) reject the hypothesis of market integration, Restoy and Rodríguez (2003) find that the evidence is more favorable to this hypothesis if a partial rather than a general equilibrium approach is employed.

Finally, Ayuso and Restoy (2004) extend the analysis in this paper to a two-good setting (consumption and housing services) and derive an approximate equilibrium relation between house prices and rents. They find that house price

dynamics can be reasonably well represented in the US, the UK and Spain by a partial adjustment model in which house prices slowly adjust to the approximate equilibrium path.

. C

We have shown in this paper that the equilibrium capital asset pricing model that emerges from Kreps-Porteus GIP preferences can be written—both in the case of homoskedastic and in the case of $AR(1)$ -GARCH($1, 1$) consumption—as a generalized CCAPM in which both aversion to risk and to intertemporal substitution matter for excess returns. This generalized CCAPM features, relative to the standard CCAPM, an extra term that captures the effects on excess returns of a possible correlation between an asset return and news about future consumption, and that reflects the interaction between precautionary saving and consumption smoothing. Because of the presence of this extra term, the predictions of this generalized CCAPM can be quite different from and richer than those of the standard CCAPM. For instance, the equilibrium excess return on an asset whose return is positively correlated with consumption might well be negative.

A second contribution is that we have derived approximate equilibrium asset pricing formulas that can be used to price explicitly any asset solely as a function of its dividend process and of consumption. In particular, these formulas make it possible to compute, albeit approximately, equity returns—as distinct from the rate of return of a claim to aggregate consumption that is computed in most of the asset pricing literature— and the otherwise unobservable rate of return on wealth from consumption data alone. This method can be applied empirically to characterize the true implications of the SCAPM when the rate of return on wealth is inferred from consumption data instead of being measured as the rate of return on the stock market. Our empirical exploration suggests that stock market returns and model-based equilibrium returns on wealth behave rather differently.

Third, our paper clarifies the often forgotten role of temporal risk aversion for equilibrium asset prices: excess returns are in general not zero, and the yield curve for real bond returns is not flat, when the consumers are neutral towards timeless risks.

Finally, this paper should be viewed as our contribution to a branch of literature that attempts, through approximations, to provide an analytic understanding of the workings of models that usually must be solved numerically. This approach makes it possible to unify theoretical results and numerical insights.

See, for instance, Kimball () or Campbell ().

$$A = C + \frac{j}{T} P + B -$$

Using the lognormality assumption, we can write

$$\begin{aligned} j r_t(j) &= -j\theta \log \beta + \rho\theta E_t \sum_{k=1}^j x_{t+k} - (\theta - 1) E_t \sum_{k=1}^j r_{w,t+k} \\ &- \frac{1}{2} \left[\rho\theta^2 \text{Var}_t \left(\sum_{k=1}^j x_{t+k} \right) + (\theta - 1)^2 \text{Var}_t \left(\sum_{k=1}^j r_{w,t+k} \right) \right. \\ &\left. + 2\rho(\theta - 1) \text{Cov}_t \left(\sum_{k=1}^j x_{t+k}, \sum_{k=1}^j r_{w,t+k} \right) \right]. \end{aligned} \quad (\text{A. })$$

Similarly, under the lognormality assumption, equations (A.) and (.) yields

$$\sum_{k=0}^{j-1} E_t r_{t+k}(1) = j r_t(j) + TP(j),$$

where

$$\begin{aligned} TP(j) &= \frac{1}{2} \text{Var}_t \left[\sum_{k=0}^{j-1} r_{t+k}(1) \right] + \rho\theta \text{Cov}_t \left[\sum_{k=0}^{j-1} r_{t+k}(1), \sum_{k=1}^j x_{t+k} \right] \\ &- (\theta - 1) \text{Cov}_t \left[\sum_{k=0}^{j-1} r_{t+k}(1), \sum_{k=1}^j r_{w,t+k} \right] \end{aligned} \quad (\text{A. })$$

is the j -period term premium.

For the homoskedastic AR() process given in (.), (.) and (.) specialize to

$$r_{w,t+1} = u + \rho x_{t+1} + \frac{1 - \rho}{1 - \delta b} S_{t+1} x_{t+1}, \quad (\text{A. })$$

where

$$u = -\log \beta - \frac{\theta(1 - \rho)^2}{2} \frac{1}{(1 - \delta b)^2} \sigma_{cc}. \quad (\text{A. })$$

Equation A.) implies that

$$r_{w,t+j} - E_t r_{w,t+j} = \left(\rho + \frac{1 - \rho}{1 - \delta b} \right) (x_{t+j} - E_t x_{t+j}). \quad (\text{A. })$$

Therefore,

$$\mathbb{E}_t \sum_{k=1}^j r_{w,t+k} = uj + \rho \mathbb{E}_t \sum_{k=1}^j x_{t+k}, \quad (\text{A. })$$

$$\text{Var}_t \left(\sum_{k=1}^j r_{w,t+k} \right) = \left(\rho + \frac{1-\rho}{1-\delta b} \right)^2 \text{Var}_t \left(\sum_{k=1}^j x_{t+k} \right), \quad (\text{A. })$$

$$\text{Cov}_t \left(\sum_{k=1}^j x_{t+k}, \sum_{k=1}^j r_{w,t+k} \right) = \left(\rho + \frac{1-\rho}{1-\delta b} \right) \text{Var}_t \left(\sum_{k=1}^j x_{t+k} \right). \quad (\text{A. })$$

Now,

$$\begin{aligned} \mathbb{E}_t \sum_{k=1}^j x_{t+k} &= \mathbb{E}_t \sum_{k=1}^j \left[a(1+b+\dots+b^{k-1}) + b^k x_t \right] \\ &= j[S(a, b, j) + T(b, j)x_t], \end{aligned} \quad (\text{A. })$$

where

$$S(a, b, j) \equiv \frac{1}{j} \frac{a}{1-b} \left[j - b \frac{1-b^j}{1-b} \right] \quad \text{and} \quad T(b, j) \equiv \frac{b}{j} \frac{1-b^j}{1-b}.$$

When consumption growth is i.i.d. ($b = 0$), $S(a, 0, j) = a$ and $T(0, j) = 0$, while for one-period bonds ($j = 1$), $S(a, b, 1) = a$ and $T(b, 1) = b$.

Moreover,

$$\text{Var}_t \left(\sum_{k=1}^j x_{t+k} \right) = \sum_{k=1}^j \text{Var}_t(x_{t+k}) + 2 \sum_{k=1}^{j-1} \sum_{l=k+1}^j \text{Cov}_t(x_{t+k}, x_{t+l}). \quad (\text{A. })$$

But

$$\text{Var}_t(x_{t+k}) = \text{Var}_t \left(b^k x_t + \sum_{s=0}^{k-1} b^s \epsilon_{c,t+k-s} \right) = \frac{1-b^{2k}}{1-b^2} \sigma_{cc}, \quad (\text{A. })$$

and, for $l > k$,

$$\begin{aligned} \text{Cov}_t(x_{t+k}, x_{t+l}) &= \text{Cov}_t \left(x_{t+k}, b^{l-k} x_{t+k} \sum_{s=0}^{l-k-1} b^s \epsilon_{c,t+l-s} \right) \\ &= b^{l-k} \text{Var}_t(x_{t+k}) \\ &= b^{l-k} \frac{1-b^{2k}}{1-b^2} \sigma_{cc}. \end{aligned} \quad (\text{A. })$$

Therefore,

$$\begin{aligned}
 \text{Var}_t \left(\sum_{k=1}^j x_{t+k} \right) &= \sum_{k=1}^j \frac{1-b^{2k}}{1-b^2} \sigma_{cc} + 2 \sum_{k=1}^{j-1} \sum_{l=k+1}^j b^{l-k} \frac{1-b^{2k}}{1-b^2} \sigma_{cc} \\
 &= \sum_{k=1}^j \frac{1-b^{2k}}{1-b^2} \sigma_{cc} + 2 \sum_{k=1}^{j-1} \frac{1-b^{2k}}{1-b^2} \sum_{l=k+1}^j b^{l-k} \sigma_{cc} \\
 &= jA(b, j) \sigma_{cc}, \tag{A. }
 \end{aligned}$$

where

$$A(b, j) \equiv \frac{1}{j} \sum_{k=1}^j \frac{1-b^{2k}}{1-b^2} \left[1 + 2b \frac{1-b^{j-k}}{1-b} \right].$$

When consumption growth is i.i.d. ($b = 0$), $A(0, j) = 1$, while for one-period bonds ($j = 1$), $A(b, 1) = 1$.

Substituting (A.), (A.), (A.), (A.) and (A.) into the Euler equation for j -period bonds (A.), using (A.) and rearranging, one obtains

$$\begin{aligned}
 r_t(j) &= -\log \beta + \rho S(a, b, j) + \rho T(b, j) x_t \\
 &+ \frac{1}{2} \left\{ \frac{(\rho - \gamma)(1 - \gamma)}{(1 - \delta b)^2} - A(b, j) \left[\rho + \frac{\gamma - \rho}{1 - \delta b} \right]^2 \right\} \sigma_{cc}, \tag{A. }
 \end{aligned}$$

which is the expression for the return on a j -period bond given in (.).

Now, from equation (.) the return on a 1-period bond is

$$r_t(1) = -\log \beta + \rho a + \rho b x_t + M,$$

where

$$M = \frac{1}{2} \left\{ \frac{(\rho - \gamma)(1 - \gamma)}{(1 - \delta b)^2} - \left(\rho + \frac{\gamma - \rho}{1 - \delta b} \right)^2 \right\} \sigma_{cc}.$$

Then, from equations (A.), (A.), and (A.) the j -period term premium can be written as

$$\begin{aligned}
 TP(j) &= \rho b \left\{ -\frac{1}{2} \rho b + \rho \theta - (\theta - 1) \left(\rho + \frac{1 - \rho}{1 - \delta b} \right) \right\} \text{Var}_t \left(\sum_{k=1}^{j-1} x_{t+k} \right) \\
 &+ \rho b \left\{ \rho \theta - (\theta - 1) \left(\rho + \frac{1 - \rho}{1 - \delta b} \right) \right\} \text{Cov}_t \left(x_{t+j}, \sum_{k=1}^{j-1} x_{t+k} \right). \tag{A. }
 \end{aligned}$$

Then, using equations (A.) and (A.) and rearranging,

$$\begin{aligned}
 TP(j) &= \rho b \left[-\frac{1}{2} \rho b + \rho + \frac{\gamma - \rho}{1 - \delta b} \right] A(b, j-1)(j-1) \\
 &+ \rho b \left[\rho + \frac{\gamma - \rho}{1 - \delta b} \right] \sum_{k=1}^{j-1} b^{j-k} \frac{1-b^{2k}}{1-b^2}, \tag{A. }
 \end{aligned}$$

which coincides with expression (.) in the text.

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Table : Correlation between observed and equilibrium returns

Model	Inverse of elasticity of intertemp. substitution (ρ)						
	.	.	-.	-.	-.	-.	-.
Consumption ($r_e = r_w$)	.	.	-.	-.	-.	-.	-.
Output ($r_e = r_w$)	.	-.	-.	-.	-.	-.	-.
Dividends ($r_e \neq r_w$)	-.

Equilibrium returns are generated according to expression (.), for the consumption and output models, and (.) for the dividends model. Data sources are described in the text.